ON COMBINATORIAL IDENTITIES OF ENGBERS AND STOCKER

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Abstract. We extend two combinatorial identities published by Engbers and Stocker in 2016. Among others, we prove that if $b$, $n$, and $r$ are integers such that $b \geq 1$ and $n - 1 \geq r \geq 0$, then

$$
\sum_{k=0}^{r} \binom{r}{k}^{2} \binom{k + n}{2r + b} = \sum_{k=0}^{n-1} \binom{k}{r}^{2} \binom{n - k}{b - 1}.
$$

The special case $b = 1$ is due to Engbers and Stocker.

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1. Introduction and Statement of Results

The work on this note has been inspired by an interesting research paper published by Engbers and Stocker [1] in 2016. The authors use combinatorial techniques to show that the identities

$$
\sum_{k=0}^{r} \binom{r}{k}^{2} \binom{k + n}{2r + 1} = \sum_{k=r}^{n-1} \binom{k}{r}^{2} \tag{1}
$$

and

$$
\sum_{k=r}^{2r} \binom{2(k - r)}{k - r} \binom{k}{2r - k} \binom{n}{k + 1} = \sum_{k=r}^{n-1} \binom{k}{r}^{2} \tag{2}
$$

are valid for all integers $n$ and $r$ with $n - 1 \geq r \geq 0$. Actually, they prove a bit more. They present summation formulas involving $\sum_{k=r}^{n-1} \binom{k}{r}^{s}$, where $s$ is a natural number. The identities (1) and (2) turn out to be the most attractive special cases.

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Here, we provide a different kind of extension. We study the sums
\[ S_{r,n}(b) = \sum_{k=0}^{r} \binom{r}{k} \binom{k+n}{2r+b} \]
and
\[ T_{r,n}(b) = \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \binom{n}{k+b}, \]
where \( b \) is an integer. In the next sections we use the concept of generating functions to prove new extensions of (1) and (2). Our extension of (1) reads as follows.

**Theorem 1.** Let \( b, n \) and \( r \) be integers with \( n - 1 \geq r \geq 0 \).

(i) If \( b \geq 1 \), then
\[ S_{r,n}(b) = \sum_{k=r}^{n-1} \binom{k}{r} \binom{n-k-1}{b-1}. \]

(ii) If \( b \leq 0 \), then
\[ S_{r,n}(b) = \sum_{k=0}^{-b} (-1)^{b-k} \binom{k+n}{r} \binom{-b}{k}. \]

The case \( b = 1 \) gives (1) whereas the special cases \( b = 0 \) and \( b = -1 \) lead to the elegant identities
\[ \sum_{k=0}^{r} \binom{r}{k} \binom{k+n}{2r} = \binom{n}{r}^2 \] (3)
and
\[ \sum_{k=0}^{r} \binom{r}{k} \binom{k+n}{2r-1} = \binom{n+1}{r}^2 - \binom{n}{r}^2. \] (4)

The sum
\[ U_{r,n}(b) = \sum_{k=1}^{r} \binom{r}{k} \binom{k+n}{2r+b} \]
is closely related to \( S_{r,n}(b) \). We apply
\[ 2 \binom{r}{k-1} \binom{r}{k} = \binom{r+1}{k}^2 - \binom{r}{k-1}^2 - \binom{r}{k}^2 \]
and obtain the representation
\[ U_{r,n}(b) = \frac{1}{2} \left( S_{r+1,n}(b-2) - S_{r,n+1}(b) - S_{r,n}(b) \right). \] (5)
Using (5) with \( b = 1, 0, -1 \), respectively, we conclude from Theorem 1 that the following counterparts of (1), (3) and (4) are valid:

\[
\sum_{k=1}^{r} \binom{r}{k-1} \binom{r}{k} \frac{k+n}{2r+1} = \binom{n}{r} \binom{n}{r+1} - \sum_{k=r}^{n-1} \binom{k}{r}^2,
\]

\[
\sum_{k=1}^{r} \binom{r}{k-1} \binom{r}{k} \frac{k+n}{2r} = \binom{n}{r-1} \binom{n+1}{r+1},
\]

\[
\sum_{k=1}^{r} \binom{r}{k-1} \binom{r}{k} \frac{k+n}{2r-1} = \binom{n+1}{r-1} \binom{n+2}{r+1} - \binom{n}{r-1} \binom{n+1}{r+1}.
\]

In Section 3, we prove the following generalization of (2).

**Theorem 2.** Let \( b, n \) and \( r \) be integers with \( n-1 \geq r \geq 0 \).

(i) If \( b \geq 1 \), then

\[
T_{r,n}(b) = \sum_{k=r}^{n-1} \binom{k}{r}^2 \binom{n-k-1}{b-1}.
\]

(ii) If \( b \leq 0 \), then

\[
T_{r,n}(b) = \sum_{k=0}^{-b} (-1)^{-k} \binom{k+n}{r}^2 \binom{-b}{k}.
\]

In particular, the special cases \( b = 1 \) and \( b = 0, -1 \) lead to (2) and

\[
\sum_{k=r}^{2r} \binom{2(k-r)}{k} \binom{k}{2r-k} \binom{n}{k} = \binom{n}{r}^2,
\]

\[
\sum_{k=r}^{2r} \binom{2(k-r)}{k} \binom{k}{2r-k} \binom{n}{k-1} = \binom{n+1}{r}^2 - \binom{n}{r}^2,
\]

respectively.

### 2. Proof of Theorem 1

We define

\[
F_b(x, u) = \sum_{n,r \geq 0} S_{r,n}(b) x^n u^r
\]

and

\[
q_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k.
\]
Then, see [3, pp. 78, 81],

\[ \sum_{n \geq 0} u^n q_n(x) = \frac{1}{\sqrt{1 - 2(1 + x)u + (1 - x)^2 u^2}}. \]

We obtain

\[ \sum_{n \geq 0} x^n \sum_{k=0}^r \binom{r}{k}^2 \binom{k+n}{2r} = \sum_{k=0}^r \binom{r}{k}^2 \frac{x^{2r-k}}{(1-x)^{2r+1}} = \frac{x^r}{(1-x)^{2r+1}} q_r(x) \]

and furthermore

\[ F_0(x, u) = \sum_{r \geq 0} u^r \sum_{n \geq 0} x^n \sum_{k=0}^r \binom{r}{k}^2 \binom{k+n}{2r} = \frac{1}{1-x} \sum_{r \geq 0} \left( \frac{u x}{(1-x)^2} \right)^r q_r(x) \]

\[ = \frac{1}{1-x} \frac{1}{\sqrt{1 - 2(1 + x) \frac{u x}{(1-x)^2} + (1 - x)^2 \frac{u^2 x^2}{(1-x)^4}}} \]

\[ = \frac{1}{\sqrt{1 - 2(1 + u) x + (1 - u)^2 x^2}} \]

\[ = \sum_{n, r \geq 0} \binom{n}{r}^2 x^n u^r. \]

Comparing the coefficients of \( x^n u^r \) we find the identity

\[ S_{r, n}(0) = \sum_{k=0}^r \binom{r}{k} \binom{k+n}{2r} = \binom{n}{r}^2. \]

Now, let \( b \geq 1 \). Applying the following variant of the Vandermonde formula, see [2, p. 169],

\[ \binom{k+n}{2r + b} = \sum_{j=0}^{n-1} \binom{k+j}{2r} \binom{n-j-1}{b-1} \quad (0 \leq k \leq r) \]

we obtain

\[ S_{r, n}(b) = \sum_{k=0}^r \binom{r}{k}^2 \sum_{j=0}^{n-1} \binom{k+j}{2r} \binom{n-j-1}{b-1} \]

\[ = \sum_{j=0}^{n-1} \binom{n-j-1}{b-1} S_{r, j}(0) = \sum_{j=0}^{n-1} \binom{n-j-1}{b-1} \binom{j}{r}. \]
Next, let \( b \leq 0 \). Using the Vandermonde type identity, see [2, p. 169],

\[
\binom{k+n}{2r+b} = \sum_{j=0}^{-b} \binom{-b}{j} \binom{k+j+n}{2r} (-1)^{-b-j}
\]

we get

\[
S_{r,n}(b) = \sum_{k=0}^{r} \binom{r}{k} \sum_{j=0}^{-b} \binom{-b}{j} \binom{k+j+n}{2r} (-1)^{-b-j}
\]

\[
= \sum_{j=0}^{-b} \binom{-b}{j} (-1)^{-b-j} S_{r,j+n}(0) = \sum_{j=0}^{-b} \binom{-b}{j} (-1)^{-b-j} \binom{j+n}{r}^2.
\]

This completes the proof of Theorem 1.

### 3. Proof of Theorem 2

As before, we consider the bivariate generating function

\[
G_t(x,u) = \sum_{n,r \geq 0} T_{r,n}(b)x^n u^r.
\]

We have, see [3, page 73]:

\[
\sum_{n \geq 0} u^n \sum_{0 \leq k \leq n/2} \binom{2k}{k} \binom{n}{2k} x^{2k} (1-2x)^{n-2k} = \frac{1}{\sqrt{1-(1-2x)u^2}}.
\]

Next, we replace \( x \) by \( \sqrt{x}/(1+2\sqrt{x}) \) and \( u \) by \( (1+2\sqrt{x})u \). This leads to

\[
\sum_{n \geq 0} u^n \sum_{0 \leq k \leq n/2} \binom{2k}{k} \binom{n}{2k} x^k = \frac{1}{\sqrt{(1-u)^2 - 4xu^2}}.
\]

We set \( t = z/(1-z) \) and apply

\[
\sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \sum_{n \geq 0} \binom{n}{k} z^n = \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \frac{z^k}{(1-z)^{k+1}}.
\]

Then we obtain

\[
G_0(z,u) = \sum_{r \geq 0} \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \frac{z^k}{(1-z)^{k+1}}
\]

\[
= \frac{1}{1-z} \sum_{r \geq 0} \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} t^k.
\]
\[
\begin{align*}
&= \frac{1}{1-z} \sum_{k \geq 0} \sum_{r \geq 0} u^{k-r} \binom{2r}{r} \binom{k}{2r} t^k \\
&= \frac{1}{1-z} \sum_{k \geq 0} (ut)^k \sum_{r \geq 0} u^{-r} \binom{2r}{r} \binom{k}{2r} \\
&= \frac{1}{1-z} \frac{1}{\sqrt{(1-ut)^2 - 4ut^2}} \\
&= \frac{1}{\sqrt{1-2(1+u)z + (1-u)^2 z^2}} \\
&= \sum_{n,r \geq 0} \binom{n}{r}^2 z^n u^r.
\end{align*}
\]

We compare the coefficients of \(z^n u^r\) and find
\[
T_{r,n}(0) = \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \binom{n}{k} = \binom{n}{r}^2.
\]

Now, let \(b \geq 1\). Using
\[
\binom{n}{k+b} = \sum_{j=0}^{n-1} \binom{j}{k} \binom{n-j-1}{b-1}
\]
leads to
\[
T_{r,n}(b) = \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \sum_{j=0}^{n-1} \binom{j}{k} \binom{n-j-1}{b-1} \\
= \sum_{j=0}^{n-1} \binom{n-j-1}{b-1} T_{r,j}(0) = \sum_{j=0}^{n-1} \binom{n-j-1}{b-1} \binom{j}{r}^2.
\]

Next, let \(b \leq 0\). Since
\[
\binom{n}{k+b} = \sum_{j=0}^{-b} \binom{-b}{j} \binom{j+n}{k}(-1)^{-b-j},
\]
we obtain
\[
T_{r,n}(b) = \sum_{k=r}^{2r} \binom{2(k-r)}{k-r} \binom{k}{2r-k} \sum_{j=0}^{-b} \binom{-b}{j} \binom{j+n}{k}(-1)^{-b-j} \\
= \sum_{j=0}^{-b} \binom{-b}{j}(-1)^{-b-j} T_{r,j+n}(0) = \sum_{j=0}^{-b} \binom{-b}{j}(-1)^{-b-j} \binom{j+n}{r}^2.
\]
The proof of Theorem 2 is complete.

REFERENCES


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