DETERMINANTS CONTAINING RISING POWERS OF FIBONACCI NUMBERS

HELMUT PRODINGER

ABSTRACT. A matrix containing rising powers of Fibonacci numbers is investigated. The LU-decomposition is guessed and proved; this leads to a formula for the determinant. Similar results are also obtained for a matrix of Lucas numbers.

1. INTRODUCTION

Carlitz [1], motivated by earlier writings, loc. cit., computed the determinant

\[
\begin{vmatrix}
F_r^n & F_r^{n+1} & F_r^{n+2} & \cdots \\
F_r^{n+1} & F_r^{n+2} & F_r^{n+3} & \cdots \\
F_r^{n+2} & F_r^{n+3} & F_r^{n+4} & \cdots \\
\cdots & \cdots & \cdots & \ddots
\end{vmatrix},
\]

with the result

\[
(-1)^{\binom{r+1}{2}}(n+1) \prod_{j=0}^{r-1} \binom{r}{j} \cdot (F_1 F_2^{-1} \cdots F_r)^2;
\]

\(F_i\) are Fibonacci numbers as usual.

In the present note we consider the rising powers analogue

\[
M = \begin{pmatrix}
F^{(r)}_n & F^{(r)}_{n+1} & F^{(r)}_{n+2} & \cdots \\
F^{(r)}_{n+1} & F^{(r)}_{n+2} & F^{(r)}_{n+3} & \cdots \\
F^{(r)}_{n+2} & F^{(r)}_{n+3} & F^{(r)}_{n+4} & \cdots \\
\cdots & \cdots & \cdots & \ddots
\end{pmatrix}.
\]

This is an \((r + 1) \times (r + 1)\) matrix, and we assume that the indices run from 0, \ldots, \(r\). The rising products are defined as follows:

\[
F^{(r)}_m := F_m F_{m+1} \cdots F_{m+r-1}.
\]

Although this definition looks more complicated than the one used by Carlitz, it is actually nicer, since we are able to compute (first guessing, then proving) the LU-decomposition of \(M = LU\), from which the determinant is an easy corollary, via \(\det(M) = U_{0,0} U_{1,1} \cdots U_{r,r}\).
2. THE LU-DECOMPOSITION OF $M$

We start from the Binet form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} \frac{1-q^n}{1-q},$$

with

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}, \quad q = \frac{\beta}{\alpha} = -\frac{1}{\alpha^2},$$

so that $\alpha = iq^{-1/2}$. We write further

$$F_{n+j} = y\alpha^{i-1} \frac{1-xq^j}{1-q},$$

with

$$y = \alpha^n \quad \text{and} \quad x = q^n.$$

Thus

$$M_{ij} = T_{n+i+j}^{(r)} = \frac{y^r}{(1-q)^r} \alpha^{(i+j-1)r+1} (xq^{i+j}; q)_r.$$

Here, we use standard $q$-notation: $(x; q)_m := (1 - x)(1 - xq) \ldots (1 - q^{m-1})$.

This is the form that we use to guess (and then prove) the LU-decomposition. It holds for general variables $x, y, q, \alpha$. However, for our application, we will then specialize. For these specializations, we need the notation of a Fibonacci-factorial:

$$n!_F := F_1 F_2 \ldots F_n.$$

**Theorem 1.** For $0 \leq i \leq j \leq r$,

$$U_{i,j} = \frac{x^i y^r}{(1-q)^r} \alpha^{(i+j)+1} q^{\frac{3(i+1)}{2}} (-1)^i (x; q)_{i+r}(x; q)_{i-1}(q; q)_{r} (x; q)_{i+j}(x; q)_{2i-1}(q; q)_{r-i}(q; q)_{j-i}.$$

For $0 \leq j \leq i \leq r$,

$$L_{i,j} = \frac{(x; q)_{i+r}(q; q)_{r} (x; q)_{i+j}(q; q)_{j-i}}{(x; q)_{i+r}(x; q)_{i+j}(q; q)_{j-i}}\alpha^{r(i-j)}.$$

**Corollary 1.** The specialized versions (Fibonacci numbers) are as follows:

$$U_{i,j} = \frac{(n + j + r - 1)!_F (n + i - 2)!_F j!_F r!_F}{(n + i + j - 1)!_F (n + 2i - 2)!_F (r - i)!_F (j - i)!_F} (-1)^{\frac{r(i+1)}{2}+ni},$$

$$L_{i,j} = \frac{(n + i + r - 1)!_F (n + 2j - 1)!_F j!_F}{(n + j + r - 1)!_F (n + i + j - 1)!_F (n + 2i - 2)!_F (r - i)!_F}.$$

**Theorem 2.** The determinant of the matrix $M$ is given by

$$\det(M) = \prod_{i=0}^r U_{i,i} = (-1)^{\binom{r}{2} + n\binom{r+1}{2}} (r!_F)^{r+1} \prod_{i=0}^r \frac{(n + i + r - 1)!_F (n + i - 2)!_F}{(n + 2i - 1)!_F (n + 2i - 2)!_F}$$

$$= (-1)^{\binom{r}{2} + n\binom{r+1}{2}} (r!_F)^{r+1}. \square$$
Although it is not necessary for our determinant calculation, we briefly mention two additional results (first general, then specialized):

**Theorem 3.**

\[ U_{i,j}^{-1} = \frac{(q;q)_{2j}(q;q)_{r-j}(x;q)_{i+j-1}}{(q;q)_i(q;q)_r(q;q)_{j-i}(x;q)_{j-1}(x;q)_i} \times \frac{q^{-ij-\frac{(i+1)j}{2}} (-1)^j (1-q)^r \alpha^{-r(i+j)-\frac{(i-1)j}{2}} x^j y^r}{q^{\frac{(i+1)j}{2}}} \]

\[ L_{i,j}^{-1} = \frac{(x;q)_{i+j}(x;q)_{i+j-1}(q;q)_i}{(x;q)_{i+j}(x;q)_{i+j-1}(q;q)_i} \times \frac{q^{\frac{(i+1)j}{2}} \alpha^{r(i-j)} (-1)^{-j}}{q^{\frac{(i+1)j}{2}}} \]

\[ U_{i,j}^{-1} = \frac{(n+2j-1)!_F (n+i+j-2)!_F (r-j)!_F}{(n+j-2)!_F (n+i+j-1)!_F r!_F (j-1)!_F i!_F} (-1)^{j+i+j+\frac{(i-1)j}{2}} \]

\[ L_{i,j}^{-1} = \frac{(n+i+r-1)!_F (n+i+j-2)!_F i!_F}{(n+j+r-1)!_F (n+2i-2)!_F j!_F (i-j)!_F} (-1)^{i+j+\frac{(i+1)j}{2}} \]

3. **Sketch of Proof**

We must simplify the following sum:

\[ \sum_j L_{i,j} U_{j,k} = \sum_j \frac{(x;q)_{i+j}(q;q)_i(x;q)_{2j}}{(x;q)_{i+j}(x;q)_{i+j}(q;q)_i} \alpha^{r(i-j)} \]

\[ \times \frac{x^j y^r}{(1-q)^j} \alpha^{r(j+k)+\frac{(j+1)k}{2}} q^{\frac{3(i-1)}{2}} (-1)^j \]

\[ \times \frac{(x;q)_{k+j}(x;q)_{j-1}(q;q)_k(q;q)_r}{(x;q)_{k+j}(x;q)_{k+j} q^{j-i-j}(q;q)_k} \]

Apart from constant factors, we are left to compute

\[ \sum_{j=0}^{\min\{i,k\}} x^j (-1)^j q^{\frac{3(i-1)j}{2}} \]

\[ \frac{(x;q)_{2j}(x;q)_{j-1}}{(x;q)_{i+j}(x;q)_{i+j}(x;q)_{j+k}(x;q)_{j+k-1}(q;q)_j(q;q)_{i-j}(q;q)_r(q;q)_{k-j}} \cdot \]

Zeilberger’s algorithm [2] (the q-version of it) readily evaluates this as

\[ \frac{(x;q)_{i+k+r}}{(x;q)_{i+k+r}(x;q)_{k+r}(x;q)_{i+k+r}(q;q)_{i+k+r}(q;q)_k(q;q)_{i+k+r}} \cdot \]

Putting this together with the constant factors, this proves that \( LU = M \).
4. The Lucas matrix

We briefly discuss the case of the matrix $\mathcal{M}$, where each $F_i$ is replaced by the Lucas number $L_i$. We also need the notation $m!_L := L_1L_2\ldots L_m$.

We write $L_m = \alpha^m + \beta^m = \alpha^n(1 + q^n)$ and $L_{n+j} = y\alpha^j(1 + xq^j)$, with $y = \alpha^n$ and $x = q^n$, when it comes to specializations.

**Theorem 4.** The LU-decomposition $\mathcal{M} = \mathcal{L}\mathcal{U}$ is given by:

\[
\mathcal{U}_{i,j} = \frac{(-x;q)_{i+j}(-x;q)_{i-1}(q;q)_j(q;q)_i}{(q;q)_{i-1}(-x;q)(q;q)_{i-j+1}(q;q)_{2j-1}} x^iy^r q^{\frac{3i(i+1)}{2} + \frac{r(r+1)}{2}} \alpha^{r(i+j)+\frac{r(r+1)}{2}},
\]

\[
\mathcal{L}_{i,j} = \frac{(-x;q)_{i+j}(-x;q)_{2j}q_{i-j}}{(-x;q)_{i-1}(-x;q)(q;q)_{i-j}} x^{-y} q^{-r(i-j) - \frac{j(j+1)}{2} + \frac{(r-1)(r)}{2}} \alpha^{-r(i+j) + \frac{r(r-1)}{2}},
\]

\[
\mathcal{U}^{-1}_{i,j} = \frac{(-x;q)_{i+j}(-x;q)_{i-j+1}(q;q)_j(q;q)_{i-j}}{(-x;q)_{i-1}(-x;q)(q;q)_{2j-1}} x^{-y} q^{-r(i-j) - \frac{j(j+1)}{2} + \frac{(r-1)(r)}{2}} \alpha^{r(i-j) - (i-j)}.
\]

**Theorem 5.** The specialized (Fibonacci/Lucas) forms are:

\[
\mathcal{U}_{i,j} = \frac{(n+j+r-1)\Gamma_L(n+i-2)!\Gamma_L n!_F}{(n+i+j-1)\Gamma_L(n+2i-2)!\Gamma_L (j-i)!\Gamma_L (r-i)!\Gamma_L} 5^r(-1)^{\frac{d(i+1)}{2} + ni},
\]

\[
\mathcal{L}_{i,j} = \frac{(n+i+r-1)\Gamma_L(n+2j-1)!\Gamma_L}{(n+i+j-1)\Gamma_L(n+i+j-1)!\Gamma_L i!\Gamma_L (i-j)!\Gamma_L} 5^r(-1)^{\frac{d(i+1)}{2} + (n+1)j},
\]

\[
\mathcal{U}^{-1}_{i,j} = \frac{(n+j+r-1)\Gamma_L(n+i+j-2)!\Gamma_L r!\Gamma_L (j-i)!\Gamma_L}{(n+j-1)\Gamma_L(n+i+r-1)!\Gamma_L (r-j)!\Gamma_L} 5^r(-1)^{\frac{d(i+1)}{2} + \frac{(r-1)}{2}},
\]

\[
\mathcal{L}^{-1}_{i,j} = \frac{(n+i+r-1)\Gamma_L(n+i+j-2)!\Gamma_L}{(n+j+r-1)\Gamma_L(n+2i-2)!\Gamma_L (i-j)!\Gamma_L} (-1)^{\frac{d(i+1)}{2} + \frac{(r-1)}{2}}.
\]

**Theorem 6.** The determinant of the matrix $\mathcal{M}$ is given by

\[
\det(\mathcal{M}) = \prod_{i=0}^{r} U_{i,i} = \prod_{i=0}^{r} \frac{(n+i+r-1)\Gamma_L(n+i-2)!\Gamma_L r!\Gamma_L 5^r(-1)^{\frac{d(i+1)}{2} + ni}}{(n+2i-1)!\Gamma_L(n+2i-2)!\Gamma_L (r-i)!\Gamma_L},
\]

\[
= (r!_F)^{r+1} 5^r(-1)^{\frac{(r^2)}{3} + n\binom{r}{3}}, \quad \square
\]

**References**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF STELLENBOSCH 7602, STELLENBOSCH, SOUTH AFRICA
E-mail address: hproding@sun.ac.za