1. Introduction

V. Retakh introduced the following restricted class of Dyck paths: Peaks are only allowed on level 1 and on even-numbered levels. Here is an example, and the corresponding plane tree using the standard bijection.

Ekhad and Zeilberger [3] proved a few days ago that these restricted paths are enumerated by Motzkin numbers. Recall that the generating function of the Motzkin numbers $M(z)$ according to length satisfies $M = 1 + zM + z^2M^2$ and thus

$$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}.$$
In this note, I want to present a few additional observations, also including the height of the paths (or the associated plane trees). First, we are going to confirm the connection to Motzkin paths:

Since the level 1 is somewhat special, we consider only trees as symbolized by the triangle. We will use two generating functions, to deal with the odd/even situation. We have

\[ G = \frac{z}{1-F} \quad \text{and} \quad F = \frac{zG}{1-G}. \]

Solving, we find \( G(z) = z^2 M(z) \) and the total generating function is

\[ \frac{z}{1-z} \sum_{r \geq 0} \left( \frac{G}{1-z} \right)^r = z M(z), \]

as predicted. Recall that the number of nodes in trees is always one more than the half-length of the corresponding Dyck path.

We will compute the average height of such restricted paths, using singularity analysis of generating functions, as in [4, 5].

2. The Height

Now we will use the substitution \( z = \frac{v}{1+v+y} \), which occurred for the first time in [8], but has been used more recently in different models where Motzkin numbers are involved [9, 7, 1], although within different models. For example, simply \( M(z) = 1 + z + z^2 \). We define

\[ F_{k+1} = \frac{z}{1 - \frac{zF_k}{1-F_k}}, \quad \text{with} \quad F_1 = z. \]

There is a simple formula, viz.

\[ F_k = \frac{v}{1+v} \left( 1 - \frac{1}{v^{2k+1}} \right). \]

This is easy to prove by induction. And then

\[ G_k = \frac{zF_k}{1-F_k} = \frac{v^2}{1+v} \frac{1-v^{2k}}{1+v^2 1-v^{2k+2}}. \]

For \( k \geq 1 \), \( G_k \) is the generating function of trees (like in the triangle) of height \( \leq 2k \).

Note that the height is currently counted in terms of nodes;

\[ G_1 = \frac{z^2}{1-z}, \]

which describes a root with \( l \geq 1 \) leaves attached to the root.

Now we incorporate the irregular beginning of the tree and compute

\[ \frac{z}{1-z} \sum_{r \geq 0} \left( \frac{G_h}{1-z} \right)^r = \frac{z}{1-z} \frac{1}{1 - \frac{G_h}{1-z}} = \frac{1-v^{2h+2}}{1-v^{2h+4}}. \]
From here onwards it seems to be more natural to define the height of the whole tree in terms of the number of edges, and then the quantity we just derived is the generating function of all trees with height $\leq 2h$, for $h \geq 1$. Note that the limit $h \to \infty$ gives us simply $v = zM(z)$, which is consistent. There is also a contribution of trees of height $\leq 1$, namely $\frac{z}{1-z} = \frac{v}{1+v^2}$, but this term is, when we compute the average height, irrelevant and only contributes to the error term, as we only compute only the leading term, which is of order $\sqrt{n}$.

So, apart from normalization, we are led to investigate

$$\sum_{h \geq 1} 2h \left[ \frac{v}{1-v^{2h+2}} - \frac{v^{2h+2}}{1-v^{2h+2}} \right] = 2v(1-v^2) \sum_{h \geq 1} h \left[ \frac{v^{2h+4}}{1-v^{2h+4}} - \frac{v^{2h+2}}{1-v^{2h+2}} \right]$$

$$= 2v(1-v^2) \sum_{h \geq 0} h \frac{v^{2h+4}}{1-v^{2h+4}} - 2v(1-v^2) \sum_{h \geq 0} (h+1) \frac{v^{2h+4}}{1-v^{2h+4}}$$

$$= -2v + 2(1-v^2) \sum_{h \geq 1} \frac{v^{2h}}{1-v^{2h}}.$$

Note that we could get explicit coefficients form here, using trinomial coefficient, $\binom{n,3}{k} = [v^k](1+v+v^2)^n$ (notation from [2]) This has to be expanded around $v = 1$, which is a standard application of the Mellin transform. Details are worked out in [6], for example:

$$\sum_{h \geq 1} \frac{v^{2h}}{1-v^{2h}} = \sum_{k \geq 1} d(k) v^{2k} \sim -\log(1-v^2) \sim -\frac{\log(1-v)}{2(1-v)}.$$

Note that $d(k)$ is the number of divisors of $k$. Consequently

$$-2v + \frac{2(1-v^2)}{v} \sum_{h \geq 1} \frac{v^{2h}}{1-v^{2h}} \sim -2\log(1-v).$$

We have $1-v \sim \sqrt{3} \sqrt{1-3z}$, and $z = \frac{1}{3}$ is the relevant singularity when discussing Motzkin numbers. We can continue

$$-2v + \frac{2(1-v^2)}{v} \sum_{h \geq 1} \frac{v^{2h}}{1-v^{2h}} \sim -\log(1-3z).$$

The coefficient of $z^n$ in this is $\frac{3^n}{n}$. This has to be divided by

$$[z^n]zM(z) = [z^{n-1}]M(z) \sim \frac{3^{n+\frac{1}{2}}}{2\sqrt{\pi n^{3/2}}},$$

with the final result for the average height of the restricted Dyck paths:

$$\sim 2\sqrt{\frac{\pi n}{3}}.$$

Recall [8] that the average height of Motzkin paths of length $n$ is asymptotic to $\sqrt{\frac{\pi n}{3}}$. 

$$\frac{\pi n}{3}.$$
REFERENCES


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